Confidence and Credibility Intervals for the Difference of Two Proportions

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Background

- Suppose that $X_1, \cdots, X_{n_1}$ and $Y_1, \cdots, Y_{n_2}$ are two independent samples such that $X_i \sim_{iid} Bernoulli(p_1)$ and $Y_j \sim_{iid} Bernoulli(p_2)$, with $i = 1, \cdots, n_1$ and $j = 1, \cdots, n_2$.
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Frequentist intervals

- The Wald interval
  - $\hat{p}_1 - \hat{p}_2 \pm z_{1-\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$
- The adjusted Wald interval
  - $\hat{p}_1 - \hat{p}_2 \pm z_{1-\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} \pm \frac{n_1 + n_2}{2n_1 n_2}$

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  - $\tilde{p}_1 - \tilde{p}_2 \pm z_{1-\alpha/2}\sqrt{V(\tilde{p}_1, \tilde{n}_1) + V(\tilde{p}_2, \tilde{n}_2)}$
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  - $V(\tilde{p}_i, \tilde{n}_i) = \frac{1}{\tilde{n}_i}\left[\tilde{p}_i - \tilde{p}_i \frac{n_i}{\tilde{n}_i} + \frac{1}{2\tilde{n}_i}\right]$
- The Newcombe interval
  - Resolve $|\hat{p}_i - p_i| = z_{1-\alpha/2}$ for p_i and denote the solutions by l_i and u_i with $l_i < u_i$
  - $L_{low} = \hat{p}_1 - \hat{p}_2 - z_{1-\alpha/2}\sqrt{\frac{l_1(1-l_1)}{n_1} + \frac{u_2(1-u_2)}{n_2}}$
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- Prior for p_i: $Beta(a_i, b_i)$,
- Information in the sample $X = \sum_{j=1}^{n_1} x_j$ and $Y = \sum_{j=1}^{n_2} y_j$,
- Posterior for p_1 and p_2: $Beta(a_1 + X, b_1 + n_1 - X)$ and $Beta(a_2 + Y, b_2 + n_2 - Y)$ (Gelman, Carlin, Stern & Rubin, 2004)
- Simulation of posterior distribution of $p_1 - p_2$
  - The posterior distribution of p_1 and p_2 are independent
  - Simulate N values from the posterior distribution of p_1 and p_2: $p_1^{(1)}, \ldots, p_1^{(N)}$ and $p_2^{(1)}, \ldots, p_2^{(N)}$, respectively.
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- Compute the highest credibility interval of $100 \times (1 - \alpha)\,\%$ for $p_1 - p_2$ using the percentiles of the simulated values.
- We consider two noninformative priors for p_1 and p_2
  - $Beta(1, 1)$:
    - Provides the same weight for all values in the range $(0, 1)$ for each p_i with $i = 1, 2$
    - The prior distribution for $p_1 - p_2$ is a triangular distribution with vertices $(-1, 0)$, $(1, 0)$ and $(0, 1)$
  - $Beta(0{,}5, 0{,}5)$
    - Jeffreys prior. Provides extra weight to extreme values of p_i: values close to 0 and 1
    - The prior distribution of $p_1 - p_2$ is symmetric at the value 0, increasing for values in $(0, 1)$ and decreasing for values in $(-1, 0)$

Difference of Two Proportions
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Some intervals
Frequentist intervals
Bayesian intervals
Comparison criteria
True coverage probability
Expected length
Variance of length
Comparison among intervals
Conclusions

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    - The prior distribution of $p_1 - p_2$ is symmetric at the value 0, increasing for values in $(0,1)$ and decreasing for values in $(-1,0)$

Bayesian intervals

- Compute the highest credibility interval of $100 \times (1-\alpha)\,\%$ for $p_1 - p_2$ using the percentiles of the simulated values.
- We consider two noninformative priors for p_1 and p_2
  - $Beta(1,1)$:
    - Provides the same weight for all values in the range $(0,1)$ for each p_i with $i = 1,2$
    - The prior distribution for $p_1 - p_2$ is a triangular distribution with vertices $(-1,0)$, $(1,0)$ and $(0,1)$
  - $Beta(0{,}5, 0{,}5)$
    - Jeffreys prior. Provides extra weight to extreme values of p_i: values close to 0 and 1
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Comparison criteria

- True coverage probability

$$CP = E(I(X,Y,p_1,p_2)) = \sum_{x=0}^{n_1}\sum_{y=0}^{n_2} I(x,y,p_1,p_2)\binom{n_1}{x}p_1^x(1-p_1)^{n_1-x}\binom{n_2}{y}p_2^y(1-p_2)^{n_2-y}$$

- Expected length

$$l = E(U(X,Y) - L(X,Y))$$

- Variance of length

$$V = Var(U(X,Y) - L(X,Y))$$

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$$
\begin{aligned}
CP &= E(I(X, Y, p_1, p_2)) \\
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&\qquad \binom{n_2}{y} p_2^y (1-p_2)^{n_2-y}
\end{aligned}
$$

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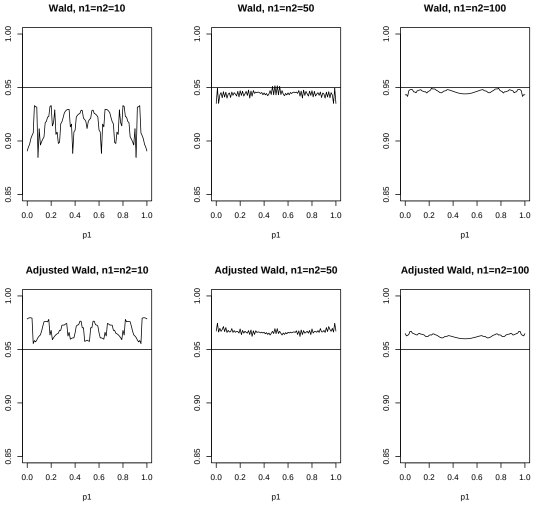


Figura: True coverage probability of the Wald and Adjusted Wald

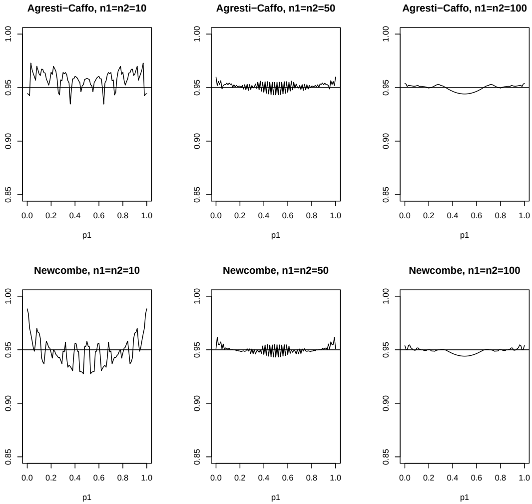
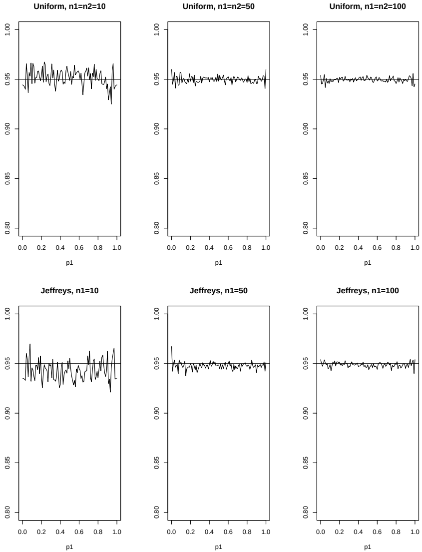


Figura: True coverage probability of the Agresti-Caffo and Newcombe


Uniform, n1=n2=10
Uniform, n1=n2=50
Uniform, n1=n2=100
Jeffreys, n1=10
Jeffreys, n1=50
Jeffreys, n1=100
0.80
0.85
0.90
0.95
1.00
0.0
0.2
0.4
0.6
0.8
1.0
p1

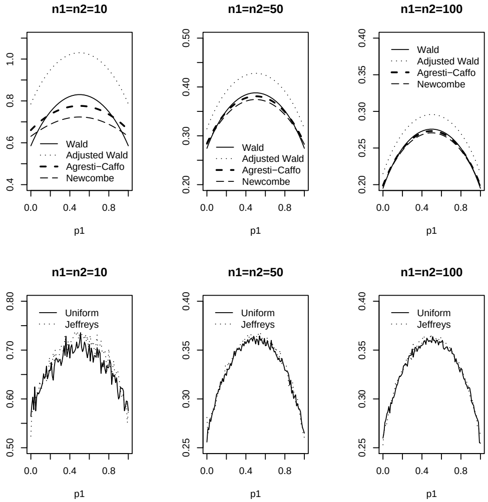


Figura: Expected length.

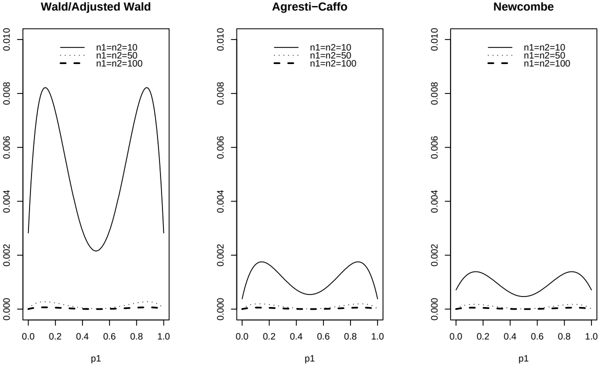
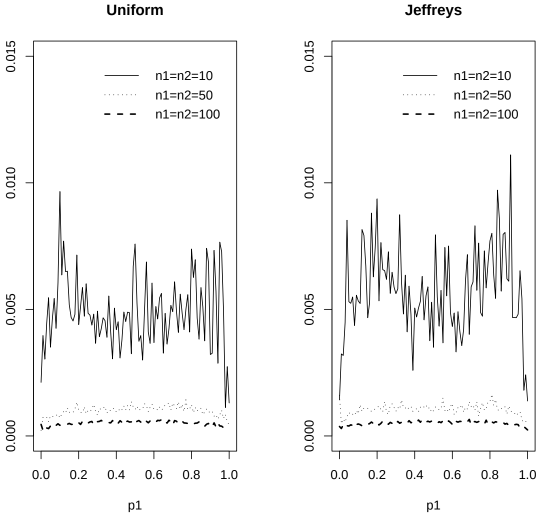


Figura: Variance of the length.


Uniform
Jeffreys
n1=n2=10
n1=n2=50
n1=n2=100
0.000
0.005
0.010
0.015
0.0
0.2
0.4
0.6
0.8
1.0
p1

Conclusions

- The performance of the bayesian intervals is not affected greatly by the sample sizes nor by different values of p_1, p_2.
- In terms of true coverage probability:
  - The best interval is the bayesian interval since its coverage probability is always close to the nominal coverage probability and stable with respect to different samples sizes.
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  - Discard the adjusted Wald interval since its large coverage probability is due to its large length.
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Muchas gracias.